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Numerical Approximation of the Solutions of Delay Differential Equations on an Infinite Interval Using Piecewise Constant Arguments

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1. INTRODUCTION

The idea of approximating the solutions of delay differential equations by equations with piecewise constant arguments (EPCA) has been suggested by Györi [1], who proved convergence of the method for linear and nonlinear delay equations on compact intervals, and under certain conditions also on the half line. The approximating equations with piecewise constant argument can, in turn, be solved by use of difference equations. The latter then also provide an approximation of the original delay equation. (The theory of EPCA was initiated and studied by Cooke and Wiener in [2] and [3].)

In this paper, we begin to investigate the question of whether in this method of approximation the asymptotic dynamics of the delay differential equations are preserved. A number of authors have dealt with the problem of showing that discretization of ordinary differential equations does not significantly alter the basic qualitative features. Readers may refer to the papers of Kloeden and Lorenz [4] and Beyn [5] for some of this work. For delay differential equations, questions of this sort have been studied by Cryer [6], Barwell [7], Zennaro [8], and later authors. The paper of Strehmel *et al.* [9] contains an up-to-date list of references.

In our method of approximation, the delay equation is first replaced by an EPCA, and then by a difference equation, and our objective is to relate the qualitative dynamics of these three equations. We obtain results for autonomous and nonautonomous equations with one or many delays. For autonomous equations, the resulting difference algorithm is just a simple Euler scheme, but for nonautonomous equations, it includes other possibilities.

In order to make the method and the nature of results as clear as possible, we begin with a particular test equation,

$$\dot{x}(t) = -px(t - \tau), \quad p > 0, \quad \tau > 0.$$

The exact region of stability of the trivial solution of this equation is given [10] by the inequality $p\tau < \frac{\pi}{2}$. We show here that if this inequality is satisfied, then for every sufficiently small step

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size h , and for every continuous initial function ϕ on $[-\tau, 0]$, the solution of the delay differential equation is uniformly approximated by the solution of the equation with piecewise constant arguments on the infinite interval $[0, \infty)$ and the exponential decay rate of the error is shown. Moreover, the supremum of the difference between these solutions tends to zero as h tends to zero. If the initial function is continuously differentiable and satisfies a compatibility condition, then the approximation is $O(h)$ on $[0, \infty)$.

In Section 2, we present an asymptotic stability result for a simple linear difference equation that will be used in subsequent sections. Section 3 contains a description of the method of approximation of the delay differential equation by an equation with piecewise constant argument. It also includes precise statements of the approximation results, Theorems 2 and 3 and corollaries, and proofs of these.

In Section 3, we show the generalizations of our results for the scalar delay differential equation

$$\dot{x}(t) = \sum_{i=0}^N P_i x(t - \tau_i),$$

with several delays and for the delay differential equation

$$\dot{x}(t) = -p(t)x(t - \tau),$$

with time dependent coefficient. As we remark at the end of the paper, our results can easily be generalized to nonautonomous systems with several delays.

2. STABILITY OF A DIFFERENCE EQUATION

For $\{f(n)\}$, let $\Delta f(n) = f(n+1) - f(n)$, $n \geq 0$.

THEOREM 1. *Consider the delay difference equation*

$$\Delta a(n) = -qa(n-k), \quad \text{for } n \geq 0, \quad (1)$$

where $q > 0$ is a real number and $k \geq 1$ is an integer such that

$$qk < 1. \quad (2)$$

Then every solution $\{x(n)\}$ of (1) satisfies

$$\lim_{n \rightarrow +\infty} x(n) = 0. \quad (3)$$

PROOF. Let $\{a(n)\}$ be a solution of Eq. (1). Then

$$a(n) - a(n-k) = \sum_{j=n-k}^{n-1} (a(j+1) - a(j)) = \sum_{j=n-k}^{n-1} \Delta a(j), \quad n \geq 0,$$

and from that and (2) it follows that

$$\begin{aligned} \Delta a(n) &= -qa(n) + q(a(n) - a(n-k)) \\ &= -qa(n) - q \sum_{j=n-k}^{n-1} \Delta a(j) \\ &= -qa(n) - q^2 \sum_{j=n-k}^{n-1} a(j-k). \end{aligned}$$

By using the definition of $\Delta a(n)$, we find

$$a(n+1) = (1-q)a(n) - q^2 \sum_{i=n-2k}^{n-k-1} a(i), \quad \text{for } n \geq k. \quad (4)$$

Set

$$M = \max_{-k \leq i \leq k} |a(i)|.$$

In that case,

$$|a(n)| \leq M, \quad (5)$$

for all $n = -k, \dots, k$. We now show that (5) is valid for all $n \geq -k$. By mathematical induction, it is enough to show that if (4) is valid for an index $n = -k, \dots, n_1$, ($n_1 \geq k$), then it is valid for $n_1 + 1$. But in that case (4) and (2) yield

$$\begin{aligned} |a(n_1+1)| &\leq (1-q)|a(n_1)| + q^2 \sum_{i=n_1-2k}^{n_1-k-1} |a(i)| \\ &\leq (1-q)M + q^2 k M = (1-q + q^2 k)M = (1-q(1-qk))M \leq M. \end{aligned}$$

Therefore, (5) is satisfied for all $n \geq -k$ and

$$M_0 = \lim_{n \rightarrow +\infty} \sup |a(n)| < \infty.$$

Thus, there exists a sequence $\{n_\ell\}$ such that $n_\ell \rightarrow +\infty$ as $\ell \rightarrow +\infty$ and

$$\lim_{\ell \rightarrow +\infty} |a(n_\ell)| = M_0.$$

From (4) we have

$$\begin{aligned} M_0 &\leq (1-q) \lim_{n \rightarrow +\infty} \sup |a(n)| + q^2 k \lim_{n \rightarrow +\infty} \sup |a(n)| \\ &= ((1-q) + q^2 k) M_0, \end{aligned}$$

and clearly

$$0 \geq M_0(q - q^2 k) = q(1 - qk)M_0.$$

By hypothesis, $0 < qk < 1$ and, hence, $M_0 = 0$. The proof of the theorem is complete. \blacksquare

REMARK 1. Theorem 1 can be easily generalized for the case of several delays. Consider the equation

$$\Delta a(n) = - \sum_{\ell=1}^m q_\ell a(n - k_\ell), \quad \text{for } n \geq 0, \quad (6)$$

where for all $\ell = 1, \dots, m$, $q_\ell > 0$ are real numbers and $k_\ell \geq 1$ are integers such that

$$\sum_{\ell=1}^m q_\ell k_\ell < 1. \quad (7)$$

Then every solution $\{a(n)\}$ of (6) satisfies (3).

REMARK 2. Condition (2) is necessary and sufficient for the asymptotic stability of the zero solution of equation (1) if $k = 1$. In fact, if $k = 1$ then equation (1) reduces to the equation

$$\Delta a(n) = -qa(n-1), \quad \text{for } n \geq 0,$$

and its characteristic equation is given in the form

$$z^2 - z + q = 0.$$

It is known [11] that every solution of $\Delta a(n) = -qa(n-1)$ tends to zero as $n \rightarrow +\infty$, if and only if the magnitudes of the roots of the characteristic equation are less than 1. By using the linear transformation [4]

$$z = \frac{\lambda + 1}{\lambda - 1},$$

we find

$$\left(\frac{\lambda + 1}{\lambda - 1}\right)^2 - \left(\frac{\lambda + 1}{\lambda - 1}\right) + q = 0.$$

Multiplying by $(\lambda - 1)^2$, we have

$$(\lambda + 1)^2 - (\lambda + 1)(\lambda - 1) + q(\lambda - 1)^2 = 0$$

and, clearly,

$$q\lambda^2 + 2(1 - q)\lambda + 2 + q = 0, \quad (q > 0).$$

On the other hand, the magnitudes of the roots of the characteristic equation are less than 1 if and only if the preceding polynomial is stable. Applying the Routh-Hurwitz test [12], we find that it is stable if and only if $2(1 - q) > 0$ and $(2 + q) > 0$, that is $q > 1$.

3. APPROXIMATION THEOREMS

We now consider the delay differential equation,

$$\dot{x}(t) = -px(t - \tau), \quad \tau \geq 0, \quad (8)$$

with the initial condition

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \phi \in C \equiv C([- \tau, 0], R), \quad (9)$$

where $p > 0$ and $\tau > 0$ are real numbers. In [1] was introduced an approximating delay differential equation with piecewise constant argument related to equation (8) in the following form:

$$\dot{y}(t) = -py \left(\left[\frac{t}{h} - \left[\frac{\tau}{h} \right] \right) h \right), \quad t \geq 0, \quad (10)_h$$

with the initial condition

$$y(nh) = \phi(nh), \quad n = -k, \dots, 0, \quad (11)_h$$

where $h = \frac{\tau}{k}$ and $k \geq 1$ is an integer, and $[.]$ is the usual greatest integer function. In our discussion, the restriction $h = \frac{\tau}{k}$ ($k \geq 1$) on the step-length h can be dropped. In that case, the definition of k is $k = \lceil \frac{\tau}{h} \rceil$, where h is an arbitrary positive step-length and our stability results are related to the definition of GP-stability (see still Barwell [7]).

It is known [1] that the initial value problem $(10)_h$ – $(11)_h$ has exactly one solution $y_h(\phi)(t)$ on $[0, \infty]$ and

$$y_h(\phi)(t) = a(n) + \frac{a(n+1) - a(n)}{h}(t - nh), \quad t \in [nh, (n+1)h), \quad (12)$$

$$n = 0, 1, 2, \dots$$

where the sequence $a(n) = a_h(\phi)(n)$ is the solution of the discrete difference equation

$$\Delta a(n) = -pha(n - k), \quad \text{for } n \geq 0, \quad (13)_h$$

with initial condition

$$a(n) = \phi(nh), \quad n = -k, \dots, 0. \quad (14)_h$$

From (12), it is obvious that the solution $y_h(\phi)(t)$ of $(10)_h$ – $(11)_h$ tends to zero, as $t \rightarrow +\infty$, if and only if the solution $\{a_h(\phi)(n)\}$ of $(13)_h$ and $(14)_h$ tends to zero, as $n \rightarrow +\infty$.

Thus, the next result is a corollary to the above discussion and to Theorem 1, moreover to a well-known stability result for the delay equation (8) in [10]:

COROLLARY 1. *Assume that $p > 0$ and $\tau > 0$ are given real numbers such that*

$$p\tau < 1. \quad (15)$$

Then the following statements are valid:

- (a) *For each $\phi \in C$ the solution $x(\phi)(t)$ of (8) and (9) tends to zero, as $t \rightarrow +\infty$.*
- (b) *For each $\phi \in C$ and $h = \frac{\tau}{k}$ ($k \geq 1$), the solutions $y_h(\phi)(t)$ and $\{a_h(\phi)(n)\}$ of $(10)_h$ – $(11)_h$ and $(13)_h$ – $(14)_h$, respectively, tend to zero as $t \rightarrow \infty$ and $n \rightarrow +\infty$.*

For any finite interval, the following theorem gives the relationship between the solutions of (8) and the solutions of $(10)_h$. This statement is a corollary of a more general theorem in [1].

THEOREM 2. *Assume that $p \in R$ and $\tau > 0$. Then the following statements are valid:*

- (a) *For every $\phi \in C$ and $T > 0$ the solutions $x(\phi)(t)$ and $y_h(\phi)(t)$ of the initial value problems (8)–(9) and $(10)_h$ – $(11)_h$, respectively, satisfy*

$$\max_{0 \leq t \leq T} |x(\phi)(t) - y_h(\phi)(t)| \rightarrow 0 \text{ as } h \rightarrow 0. \quad (16)$$

- (b) *If $\phi \in C'_0 \equiv \{\psi \in C: \dot{\psi} \in C \text{ and } \dot{\psi}(0-) = -p\psi(-\tau)\}$ and $T > 0$ then there exists a constant $M = M(T, \phi) > 0$ such that for all $h = \frac{\tau}{k}$ ($k \geq 1$), and for all $t \in [0, T]$*

$$|x(\phi)(t) - y_h(\phi)(t)| \leq Mh. \quad (17)$$

Let $p > 0$ and $\tau > 0$ be real numbers such that

$$p\tau < \frac{\pi}{2} \quad (18)$$

or equivalently for every $\phi \in C$ the solution $x(\phi)(t)$ of (8) and (9) tends to zero, as $t \rightarrow +\infty$ [10].

Under the condition (18) we are going to study the following two problems which arose from Corollary 1 and Theorem 2:

What conditions must $h = \frac{\tau}{k}$, ($k \geq 1$), satisfy in order that

- (i) *the solutions $y_h(\phi)(t)$ and $a_h(\phi)(n)$ of $(10)_h$ – $(11)_h$ and $(13)_h$ – $(14)_h$, respectively, satisfy*

$$\lim_{t \rightarrow +\infty} y_h(\phi)(t) = 0 = \lim_{n \rightarrow +\infty} a_h(\phi)(n) \quad (19)$$

for every $\phi \in C$;

- (ii) *the solution $x(\phi)(t)$ of (8)–(9) is approximated uniformly on $[0, \infty)$ with the solution $y_h(\phi)(t)$ of $(10)_h$, that is*

$$\sup_{0 \leq t \leq \infty} |x(\phi)(t) - y_h(\phi)(t)| \rightarrow 0, \quad \text{as } h \rightarrow 0, \quad (20)$$

for every $\phi \in C$?

A corollary of the next theorem gives a relatively sharp answer to the above problems.

THEOREM 3. Assume $p > 0$ and $\tau > 0$ are real numbers such that (18) holds. Then there exist $L > 0$ and $\alpha_0 > 0$ such that the following statements are valid:

- (a) For every $h = \frac{\tau}{k}$, ($k \geq 1$), and for every $\phi \in C$ there are $M_1(\phi, h) \geq 0$ and $M_2(\phi) \geq 0$ such that $M_1(\phi, h) \rightarrow 0$, as $h \rightarrow 0$, and

$$|x(\phi)(t) - y_h(\phi)(t)| \leq (M_1(\phi, h) + M_2(\phi)ht) e^{-(\alpha_0 - Lh)t}, \quad t \geq 4\tau. \quad (21)$$

- (b) For every $\phi \in C'_0 \equiv \{\psi \in C: \dot{\psi} \in C \text{ and } \dot{\psi}(0-) = -p\psi(-\tau)\}$ there exists a constant $M_3(\phi) > 0$ such that

$$|x(\phi)(t) - y_h(\phi)(t)| \leq (M_3(\phi) + M_2(\phi)t) h e^{-(\alpha_0 - Lh)t}, \quad t \geq 0. \quad (22)$$

Note that C'_0 is a dense set in C in the sense that for all $\phi \in C$ there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset C'_0$ such that

$$|\phi_n(0) - \phi(0)| + \int_{-\tau}^0 |\phi_n(s) - \phi(s)| ds \rightarrow 0, \quad \text{as } n \rightarrow +\infty \quad (\text{see reference [13]}).$$

PROOF. (a) Let $\phi \in C$ be given and consider the solutions $x(t) = x(\phi)(t)$ and $y_h(t) = y_h(\phi)(t)$ of (8)–(9) and (10)_h–(11)_h, respectively. Then the function $\epsilon_h(t) = y_h(t) - x(t)$, ($t \geq 0$), satisfies

$$\begin{aligned} \dot{\epsilon}_h(t) &= p \left(x(t - \tau) - y_h \left(\left[\frac{t}{h} - \left[\frac{\tau}{h} \right] \right] h \right) \right) \\ &= -p\epsilon_h(t - \tau) + p(y_h(t - \tau) - y_h \left(\left[\frac{t}{h} - \left[\frac{\tau}{h} \right] \right] h \right)), \quad t \geq 4\tau. \end{aligned}$$

For all $s \geq 0$, it can be easily seen that

$$s - \tau - h \leq \left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \leq s - \tau + h, \quad (23)$$

and, clearly, for all $t \geq 4\tau$ and for all s between $\left[\frac{t}{h} - \left[\frac{\tau}{h} \right] \right] h$ and $t - \tau$, one has

$$t - 4\tau \leq t - 2\tau - 2h \leq \left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \leq t - 2\tau + 2h \leq t. \quad (24)$$

Thus, for $t \geq 4\tau$, (10)_h yields

$$\begin{aligned} y_h(t - \tau) - y_h \left(\left[\frac{t}{h} - \left[\frac{\tau}{h} \right] \right] h \right) &= -p \int_{[t/h - [\tau/h]]h}^{t-\tau} y_h \left(\left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \right) ds \\ &= -p \int_{[\frac{t}{h} - [\frac{\tau}{h}]]h}^{t-\tau} \epsilon_h \left(\left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \right) ds + f_h(t), \end{aligned}$$

where

$$f_h(t) = -p \int_{[\frac{t}{h} - [\frac{\tau}{h}]]h}^{t-\tau} x \left(\left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \right) ds, \quad t \geq 4\tau. \quad (25)$$

Thus,

$$\dot{\epsilon}_h(t) = -p\epsilon_h(t - \tau) - p^2 \int_{[\frac{t}{h} - [\frac{\tau}{h}]]h}^{t-\tau} \epsilon_h \left(\left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \right) ds + pf_h(t), \quad t \geq 4\tau. \quad (26)$$

Let ν be the fundamental solution of (8) on $[4\tau, \infty)$, that is $\nu: [3\tau, \infty) \rightarrow R$ is the unique function which satisfies the following properties:

- (α) $\nu(t)$ is continuous on $[4\tau, \infty]$ and $\dot{\nu}(t) = -p\nu(t - \tau)$, $t \geq 4\tau$, and
- (β) $\nu(t) = 0$, $3\tau \leq t < 4\tau$, and $\nu(4\tau) = 1$.

Moreover, let $z_h(t)$ be the unique solution of (8) on $[4\tau, \infty)$ with the initial condition

$$z_h(t) = \epsilon_h(t), \quad 3\tau \leq t \leq 4\tau. \quad (27)$$

Then by variation of constants formula [10, p. 75] the solution $\epsilon_h(t)$ of (26) can be written in the form

$$\epsilon_h(t) = z_h(t) - p^2 \int_{4\tau}^t \left(\nu(t-u) \int_{[u/h - \lceil \tau/h \rceil]h}^{u-\tau} \epsilon_h \left(\left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \right) ds \right) du + p \int_{4\tau}^t \nu(t-u) f_h(u) du,$$

for all $t \geq 4\tau$. Thus, by virtue of (23) and (24), we obtain

$$|\epsilon_h(t)| \leq |z_h(t)| + p^2 h \int_{4\tau}^t |\nu(t-u)| \max_{u-4\tau \leq s \leq u} |\epsilon_h(s)| du + p \int_{4\tau}^t |\nu(t-u)| |f_h(u)| du, \quad (28)$$

for all $t \geq 4\tau$. By hypothesis $p\tau < \frac{\pi}{2}$, that is equation (8) is asymptotically stable. Therefore, there are constants $\alpha > 0$ and $L_0 > 0$ such that

$$|\nu(t)| \leq L_0 e^{-\alpha_0 t}, \quad t \geq 4\tau, \quad (29)$$

and

$$|z_h(t)| \leq L_0 m_h e^{-\alpha_0 t}, \quad t \geq 4\tau, \quad (30)$$

where

$$m_h = \max_{0 \leq s \leq 4\tau} e^{\alpha_0 s} |\epsilon_h(s)|. \quad (31)$$

Moreover, there is a constant M_0 such that

$$|x(t)| \leq M_0 m_\phi e^{-\alpha_0 t}, \quad (32)$$

where

$$m_\phi = \max_{-\tau \leq s \leq 0} |\phi(s)|.$$

Combining (24), (25), and (32), we find

$$\begin{aligned} |f_h(t)| &\leq p \int_{[t/h - \lceil \tau/h \rceil]h}^{t-\tau} M_0 m_\phi e^{-\alpha_0 [s/h - \lceil \tau/h \rceil]h} ds \\ &\leq p M_0 m_\phi h e^{\alpha_0 4\tau} e^{-\alpha_0 t} \equiv k_\phi h e^{-\alpha_0 t}, \quad t \geq 4\tau, \end{aligned} \quad (33)$$

where

$$k_\phi = p M_0 m_\phi e^{\alpha_0 4\tau}. \quad (34)$$

By using (29), (30) and (33) in (28), we obtain

$$|\epsilon_h(t)| \leq L_0 m_h e^{-\alpha_0 t} + p^2 h L_0 \int_{4\tau}^t e^{-\alpha_0(t-u)} \max_{u-4\tau \leq s \leq u} |\epsilon_h(s)| du + k_\phi h L_0 p t e^{-\alpha_0 t},$$

for all $t \geq 4\tau$. Set

$$g_h(t) = \max_{0 \leq s \leq t} \{e^{\alpha_0 s} |\epsilon_h(s)|\}, \quad t \geq 0.$$

Then,

$$g_h(t) \leq \max\{1, L_0\} m_h + L_0 k_\phi h p t + p^2 h L_0 e^{\alpha_0 4\tau} \int_{4\tau}^t g_h(u) du,$$

for all $t \geq 4\tau$. Set

$$M_1(\phi, h) = \max\{1, L_0\} m_h, \quad M_2(\phi) = L_0 k_\phi p \quad \text{and} \quad L = p^2 L_0 e^{\alpha_0 4\tau}. \quad (35)$$

Then,

$$g_h(t) \leq M_1(\phi, h) + M_2(\phi) h t + L h \int_{4\tau}^t g_h(u) du, \quad t \geq 4\tau,$$

and by applying the Gronwall-Bellman inequality [10] we find

$$g_h(t) \leq (M_1(\phi, h) + M_2(\phi) h t) e^{L h t}, \quad t \geq 4\tau.$$

From the definitions of $\epsilon_h(t)$ and $g_h(t)$, it follows

$$\begin{aligned} |x(\phi)(t) - y_h(\phi)(t)| &= \epsilon_h(t) \leq e^{-\alpha_0 t} g_h(t) \\ &\leq (M_1(\phi, h) + M_2(\phi) h t) e^{-(\alpha_0 - L h) t}, \end{aligned}$$

for all $t \geq 4\tau$. Thus, (21) is satisfied. On the other hand, from (31) and (35), we have

$$M_1(\phi, h) \leq \max\{1, L_0\} e^{\alpha_0 4\tau} \max_{0 \leq s \leq 4\tau} |x(\phi)(s) - y_h(\phi)(s)|, \quad (36)$$

and by Theorem 2, we find $M_1(\phi, h) \rightarrow 0$, as $h \rightarrow 0$. The proof of statement (a) is complete.

(b) This is an easy consequence of equations (21) and (36), and Theorem 2(b). Therefore its proof is omitted. \blacksquare

COROLLARY 2. Assume that $p > 0$ and $\tau > 0$ are real numbers such that (18) holds. Then there exists a $h_0 > 0$ such that the following statements are valid:

- (a) For all $\phi \in C$ and for all $h \in (0, h_0)$ the solutions $y_h(\phi)(t)$ and $a_h(\phi)(n)$ of (10)_h–(11)_h and (13)_h–(14)_h, respectively, satisfy (19).
- (b) For all $\phi \in C$ and for all $h \in (0, h_0)$

$$K(\phi, h) = \sup_{t \geq 0} |x(\phi)(t) - y_h(\phi)(t)| < \infty \quad \text{and} \quad K(\phi, h) \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

4. SOME GENERALIZATIONS

We first consider the delay differential equation

$$\dot{x}(t) = \sum_{i=0}^N p_i x(t - \tau_i), \quad t \geq 0, \quad (37)$$

with several delays and with initial condition

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \phi \in C \equiv C([- \tau, 0], R), \quad (38)$$

where we assume that

- (i) for all $i = 0, 1, \dots, N$, p_i are real numbers and $0 = \tau_0 < \tau_1 < \dots < \tau_N \equiv \tau < \infty$;
- (ii) $\max\{\operatorname{Re} \lambda: \lambda = \sum_{i=0}^N p_i \exp(-\lambda \tau_i)\} < 0$ or equivalently the zero solution of equation (37) is asymptotically stable.

We now introduce an approximating delay differential equation with piecewise constant arguments related to equation (37) in the form

$$\dot{y}(t) = \sum_{i=0}^N p_i y \left(\left[\frac{t}{h} - \left[\frac{\tau_i}{h} \right] \right] h \right), \quad t \geq 0, \quad (39)_h$$

with the initial condition

$$y(nh) = \phi(nh), \quad n = -k, \dots, 0, \quad (40)_h$$

where $h = \frac{\tau}{k}$ and $k \geq 1$ is an integer and $[\cdot]$ denotes the greatest integer function.

Note here, the initial value problem $(39)_h$ – $(40)_h$ has exactly one solution $y_h(\phi)(t)$ on $[0, \infty)$ and it is given in (12), where now $a(n) = a_h(\phi)(n)$ is the solution of the difference equation

$$\Delta a(n) = h \sum_{i=0}^N p_i a(n - k_i), \quad \text{for } n \geq 0,$$

with the initial condition

$$a(n) = \phi(nh), \quad n = -k, \dots, 0$$

where $k_i = \left[\frac{\tau_i}{h} \right]$.

By using the same argument that was given in the proof of Theorem 3 and a more general form of Theorem 2 in [1], one can easily prove the following generalization of Theorem 3 for the case of several delays.

THEOREM 4. *Assume that (i) and (ii) are satisfied. Then there exist $L > 0$ and $\alpha_0 > 0$ such that the following statements are valid:*

- (a) *For every $h = \frac{\tau}{k}$, ($k \geq 1$), and for every $\phi \in C$ there are $M_1(\phi, h) \geq 0$ and $M_2(\phi) \geq 0$ such that $M_1(\phi, h) \rightarrow 0$, as $h \rightarrow 0$, and the solutions $x(\phi)(t)$ and $y_h(\phi)(t)$ of (37)–(38) and $(39)_h$ – $(40)_h$, respectively, satisfy (21).*
- (b) *For every $\phi \in \left\{ \psi \in C: \dot{\psi} \in C \text{ and } \dot{\psi}(0-) = \sum_{i=0}^N p_i \psi(-\tau_i) \right\}$ there exists a constant $M_3(\phi) > 0$ such that the solutions $x(\phi)(t)$ and $y_h(\phi)(t)$ of (37)–(38) and $(39)_h$ – $(40)_h$, respectively, satisfy (22).*

We now demonstrate that our method can be used to investigate the delay differential equations with time dependent coefficients, too. Although the next result is valid for the case of several delays, for the sake of simplicity we only deal with the single delay case.

Consider the delay differential equation

$$\dot{x}(t) = -p(t)x(t - \tau), \quad t \geq 0, \quad (41)$$

with the initial condition

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \phi \in C, \quad (42)$$

where

$$\tau > 0, \quad p \in C([0, \infty), R) \text{ and } p_0 = \sup_{t \geq 0} |p(t)| < \infty. \quad (43)$$

The approximating delay differential equation with piecewise constant argument related to equation (41) is given in the form

$$\dot{y}(t) = -p(t)y \left(\left[\frac{t}{h} - \left[\frac{\tau}{h} \right] \right] h \right), \quad t \geq 0, \quad (44)_h$$

with the initial condition

$$y(nh) = \phi(nh), \quad n = -k, \dots, 0, \quad (45)_h$$

where $h = \frac{\tau}{k}$ and $k \geq 1$.

Then one can easily see that the initial value problem (44)_h–(45)_h has exactly one solution $y_h(\phi)(t)$ on $[0, \infty)$ which is given in the form

$$y_h(\phi)(t) = a(n) - \int_{nh}^t p(u) du a(n-k), \quad \text{for } t \in [nh, (n+1)h), \quad n \geq 0,$$

where $a(n) = a_h(n)$ is the solution of the discrete difference equation

$$\Delta a(n) = - \int_{nh}^{(n+1)h} p(u) du a(n-k), \quad \text{for } n \geq 0,$$

with initial condition

$$a(n) = \phi(nh), \quad n = -k, \dots, 0,$$

where $h = \frac{\tau}{k}$.

We say that the zero solution of equation (41) is exponentially stable with parameters $L_0 > 0$ and $\alpha_0 > 0$ if for every $(t_0, \phi) \in [0, \infty) \times C$ the solution $x(t_0, \phi)(t)$ of (41)–(42) satisfies

$$|x(t_0, \phi)(t)| < L_0 \|\phi\| \exp(-\alpha_0(t - t_0)), \quad t \geq t_0, \quad (46)$$

where $\|\phi\| = \max_{-\tau \leq s \leq 0} |\phi(s)|$.

We now give a generalization of Theorem 3 for the time dependent case.

THEOREM 5. Assume that (43) is satisfied and the zero solution of equation (41) is exponentially stable with parameters $L_0 > 0$ and $\alpha_0 > 0$. Then there exists an $L > 0$ such that the following statements are valid:

- (a) For every $h = \frac{\tau}{k}$, ($k \geq 1$), and for every $\phi \in C$ there are $M_1(\phi, h) \geq 0$ and $M_2(\phi) \geq 0$ such that $M_1(\phi, h) \rightarrow 0$, as $h \rightarrow 0$, and the solutions $x(\phi)(t)$ and $y_h(\phi)(t)$ of (41)–(42) and (44)_h–(45)_h, respectively, satisfy the inequality (21).
- (b) For every $\phi \in \{\psi \in C: \dot{\psi} \in C \text{ and } \psi(0-) = p(0)\psi(-\tau)\}$ there exists a constant $M_3(\phi) > 0$ such that the inequality (22) is satisfied.

PROOF. (a) Let $\phi \in C$ be given and consider the solutions $x(t) = x(\phi)(t)$ and $y_h(\phi)(t)$ of (41)–(43) and (44)_h–(45)_h, respectively. Then the function $\epsilon_h(t) = y_h(t) - x(t)$, ($t \geq 0$), satisfies

$$\dot{\epsilon}_h(t) = -p(t)\epsilon_h(t - \tau) + p(t)(y_h(t - \tau) - y_h\left(\left[\frac{t}{h} - \left[\frac{\tau}{h}\right]\right]h\right)), \quad t \geq 4\tau.$$

Now using the same argument that is given in the proof of Theorem 3, one can easily see that

$$\dot{\epsilon}_h(t) = -p(t)\epsilon_h(t - \tau) - p(t) \int_{[t/h - [\tau/h]]h}^{t-\tau} p(s)\epsilon_h\left(\left[\frac{s}{h} - \left[\frac{\tau}{h}\right]\right]h\right) ds + p(t)f_h(t), \quad t \geq 4\tau, \quad (47)$$

where

$$f_h(t) = - \int_{[t/h - [\tau/h]]h}^{t-\tau} p(s)x\left(\left[\frac{s}{h} - \left[\frac{\tau}{h}\right]\right]h\right) ds, \quad t \geq 4\tau.$$

Let $\nu(t, s)$ be the fundamental solution of equation (41) on $[4\tau, \infty)$, that is $\nu: [3\tau, \infty) \times [4\tau, \infty) \rightarrow R$ is the unique function which satisfies the following properties:

- (α) For all fixed $s \geq 4\tau$ the function $\nu(t, s)$ is continuous on $4\tau \leq t \leq \infty$ and

$$\nu_t(t, s) = -p(t)\nu(t - \tau, s), \quad t \geq s \geq 4\tau,$$

and $(\beta) \nu(t, s) = 0, 3\tau \leq t \leq 4\tau$ and $\nu(s, s) = 1$ for $s \geq 4\tau$. Moreover, let $z_h(t)$ be the unique solution of (41) with the initial condition

$$z_h(t) = \epsilon_h(t), \quad 3\tau \leq t \leq 4\tau.$$

Then by variation of constant formula [10, p. 304] the solution $\epsilon_h(t)$ of (47) can be written in the form

$$\epsilon_h(t) = z_h(t) - \int_{4\tau}^t \nu(t, u) p(u) \int_{[u/h - [\tau/h]]h}^{u-\tau} p(s) \epsilon_h \left(\left[\frac{s}{h} - \left[\frac{\tau}{h} \right] \right] h \right) ds + \int_{4\tau}^t \nu(t, u) p(u) f_h(u) du,$$

for all $t \geq 4\tau$. Thus, by virtue of (23)–(24) and (43), we obtain

$$|\epsilon_h(t)| \leq |z_h(t)| + p_0^2 h \int_{4\tau}^t |\nu(t, u)| \max_{u-4\tau \leq s \leq u} |\epsilon_h(s)| du + p_0 \int_{4\tau}^t |\nu(t, u)| |f_h(u)| du, \quad (48)$$

for all $t \geq 4\tau$.

By hypothesis, the zero solution of equation (41) is asymptotically stable with parameters $L_0 > 0$ and $\alpha_0 > 0$. Therefore,

$$|\nu(t, s)| \leq L_0 e^{-\alpha_0(t-s)}, \quad t \geq s \geq 4\tau \quad (49)$$

and

$$|z_h(t)| \leq L_0 m_h e^{-\alpha_0 t}, \quad t \geq 4\tau, \quad (50)$$

where

$$m_h = \max_{0 \leq s \leq 4\tau} e^{\alpha_0 s} |\epsilon_h(s)|. \quad (51)$$

Combining (46), (48)–(51) and using the same argument that was given in the proof of Theorem 3, we find

$$|\epsilon_h(t)| \leq L_0 m_h e^{-\alpha_0 t} + p_0^2 h L_0 \int_{4\tau}^t e^{-\alpha_0(t-u)} \max_{u-4\tau \leq s \leq u} |\epsilon_h(s)| du + k_\phi h L_0 p_0 t e^{-\alpha_0 t}$$

for all $t \geq 4\tau$, where k_ϕ is a suitable constant. From the latest inequality, by applying the Gronwall-Bellman inequality and a general version of Theorem 2 in [1] and by a repetition of the rest of the proof of Theorem 3, one can show that statements (a) and (b) are valid. Therefore, these details are omitted. ■

REMARK 3. By using vector and matrix norms instead of the absolute values in the above results, one can generalize them for systems of delay differential equations with several delays.

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